Control of Missile Dispersion via Roll Rate Modulation

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Feedback control via roll rate modulation to limit cross-range dispersion of spinning missiles is investigated. The dispersion to be controlled is assumed to result from lift variations that do not average out over a roll cycle when the missile roll rate remains constant. By modulating the roll rate in a prescribed manner through feedback control, lift nonaveraging and the resulting cross-range dispersion are minimized. Two different approaches to the feedback control are investigated. In one approach, the roll rate is modulated by feedback of the state variables to prevent the occurence of lateral velocity increments that produce dispersion. In the other approach, the roll rate is modulated harmonically about a steady roll rate, and the amplitude of the harmonic variation is controlled to limit dispersion.

Nomenclature

A, B, C, D, E, F	= feedback gains
$H(t)$	= unit step function
i	$= \sqrt{-1}$
I	= pitch or yaw moment of inertia
I_x	= roll moment of inertia
J_n	= nth-order Bessel function
l_c	= roll control moment
l_x	= roll moment disturbance
L	= lift force
L_θ	= lift force derivative
m	= missile mass
m_t	= complex disturbance moment, $m_y + im_z$
Δm_t	= magnitude of disturbance moment
$\bar{m}$	= moment step
M_y, M_z	= pitch and yaw disturbances
p	= roll rate
q	= roll modulation rate
r	$= y + iz$
s	= Laplace transform variable
S	= aerodynamic reference area
t	= time
v	= y component of transverse velocity
V	= transverse velocity in cross plane, $v + iw$
ΔV	= transverse velocity increment, $\Delta v + i\,\Delta w$
w	= z component of transverse velocity
$\hat{z}$	= nondimensional ratio Δ/q; coordinate
α	= angle of attack
β	= angle of sideslip
γ	$= -m_y/m_z$
δ	= complex angle of attack, $\beta + i\alpha$
$\delta^*(t)$	= unit impulse function
$\Delta\delta$	= complex angle-of-attack increment
Δ	= sinusoidal control parameter
θ	= angle of attack (Euler angle)
τ	= nondimensional roll angle $p_0 t$; $t\sqrt{Aw_0}/2$, Eq. (51)
μ	= moment of inertia ratio, I_x/I
ν	= aerodynamic damping parameter
ν_m	= yaw moment damping parameter
ξ	= complex angle of attack, $\beta + i\alpha$
σ	$= \omega^2/p_0^2$
Φ_i	= roll angle relative to inertial reference
ω	= undamped natural pitch frequency

Subscripts

0	= initial value
+	= perturbation value

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Introduction

ROLL-LIFT dispersion of spinning missiles is caused by nonaveraging of the lift vector in the presence of moment disturbances that perturb the angle of attack or roll rate.[1] Lift nonaveraging can occur either from angle-of-attack variations at a constant lift precession rate[2] or from roll rate variations for a trimmed missile with constant lift[3] or from a combination of both lift and precession rate variations. Control of lift nonaveraging should therefore be possible by controlling either the lift variations, the lift vector precession rate, or a combination of both. The latter method was investigated to control the dispersion of an untrimmed missile in which both angle-of-attack and lift precession rate feedback were used in the control law.[4] In another investigation to control dispersion of a rolling trimmed missile, angle of attack and angle-of-attack rate feedbacks were applied to pitch and yaw control moments at a constant roll rate.[5]

In this paper we consider only roll rate modulation about a steady roll rate to control lift nonaveraging. Roll modulation complicates the analysis because the missile equations for complex angle-of-attack motion contain roll rate terms in the coefficients. The equations are linear for a constant roll rate but become nonlinear when the roll rate is time dependent. The magnitude of the roll coupling depends on the coordinate system used to describe the complex angle-of-attack motion. Both body-fixed and aeroballistic coordinates are used to investigate different feedback controls. The advantages and disadvantages of the different coordinate systems become apparent.

Analysis

Formulation

Small-angle complex angle-of-attack motion for a spinning missile is expressed in body-fixed coordinates, Eq. (1), and in aeroballistic coordinates, Eq. (2):

$$\ddot{\delta}+[\nu+ip(2-\mu)]\dot{\delta}+\{\omega^2-p^2(1-\mu)+i[\dot{p}+p(\nu-\nu_m)]\}\delta=im_t \quad (1)$$

$$\ddot{\xi}+(\nu-i\mu p)\dot{\xi}+(\omega^2-ip\nu_m)=im_t e^{i\int p\,dt} \quad (2)$$

The complex angles of attack in these two coordinate systems are related by

$$\xi=\delta e^{i\Phi} \quad (3)$$

where Φ is the roll angle in inertial space, defined by

$$\Phi=\int p\,dt \quad (4)$$

Lift nonaveraging caused by lift and/or roll variations produces a transverse velocity in the cross-plane with respect to inertial coordinates

$$V=v+iw \quad (5)$$

By considering the missile as a point mass under the action of a rotating lift force, we can write the transverse velocity as an integral of the lateral acceleration:

$$V=V(0)-\frac{L_\theta}{m}\int_0^t \delta e^{i\int p\,dt}\,dt \quad (6)$$

$$V=V(0)-\frac{L_\theta}{m}\int_0^t \xi\,dt \quad (7)$$

where Eq. (6) is in body-fixed coordinates and Eq. (7) is in aeroballistic coordinates. The lift force derivative L_θ is assumed constant over the time duration of the roll-lift variations. The roll acceleration,

$$\dot{p}=(l_x+l_c)/I_x \quad (8)$$

is the response to a disturbance moment l_x and a control moment l_c. The objective of the study is to define l_c in terms of measurable feedback parameters to minimize lift nonaveraging. In one approach the control moment is expressed as a function of state variables, and the feedbacks are determining such as to minimize the transverse velocity increments produced by pitch or yaw moment disturbances. In another approach, the roll rate is modulated harmonically about a steady-state value and the amplitude of the harmonic variation is determined to limit dispersion. Because the equations are nonlinear with nonconstant roll rate, suitable approximations must be made in order to estimate the feedback effectiveness analytically. Roll disturbances are not included for simplicity.

State Variable Feedback

In this approach, the control moment l_c is assumed to be a linear function of the state variables of the form

$$\dot{p}=\frac{l_c}{I_x}=A\,\Delta\alpha+B\Delta\beta+C\dot{\alpha}+D\dot{\beta}+E\int\alpha+F\int\beta \quad (9)$$

The equations of motion are linearized in the body-fixed coordinates by taking small perturbations δ_+ and p_+ about quasi-steady values δ_0 and p_0,

$$\delta=\delta_0+\delta_+ \quad (10)$$

$$p=p_0+p_+ \quad (11)$$

and neglecting higher-order terms. It is assumed that there is a means of sustaining a steady trim angle of attack δ_0.

The resulting linear control equations in terms of the perturbations are

$$\ddot{\delta}_++[\nu+ip_0(2-\mu)]\dot{\delta}_++[\omega^2-p_0^2(1-\mu)+ip_0(\nu-\nu_m)]\delta_++\{2p_0p_+(1-\mu)+i[\dot{p}_++p_+(\nu-\nu_m)]\}\delta_0=im_t \quad (12)$$

$$\dot{p}_+=A\alpha_++B\beta_++C\dot{\alpha}_++D\dot{\beta}_++E\int\alpha_++F\int\beta_+ \quad (13)$$

The transverse velocity is expressed as

$$V(t)=V(0)-\frac{L_\theta}{p_0 m}\int_0^t(\delta_0+\delta_+)e^{i\tau}e^{i\lambda}\,d\tau \quad (14)$$

where we have defined λ as

$$\lambda=\int_0^\tau \frac{p_+}{p_0}\,d\tau \quad (15)$$

and changed to a nondimensional roll angle variable τ defined by

$$d\tau=p_0\,dt \quad (16)$$

We assume a steady-state condition initially, defined by the conditions $p=p_0$ and $\delta=\delta_0$ at $t=0$. The transverse velocity is evaluated in terms of δ_+ and p_+ as follows: The upper limit of the integral in Eq. (14) is taken to be sufficiently large to include the time duration of the perturbations. Without loss of generality this limit can be taken as ∞, and Eq. (14) can be written in the form

$$V=V(0)-\frac{L_\theta}{p_0 m}\int_0^\infty F(\tau)e^{i\tau}\,d\tau \quad (17)$$

where

$$F(\tau)=(\delta_0+\delta_+)e^{i\lambda(\tau)} \quad (18)$$

From the definition of the Laplace transform,

$$F(s)=\mathscr{L}[F(\tau)]\equiv\int_0^\infty F(\tau)e^{-s\tau}\,d\tau \quad (19)$$

the integral in Eq. (17) is observed to be the Laplace transform of $F(\tau)$ with $s=-i$, or

$$V=V(0)-(L_\theta/p_0 m)[F(s)]_{s=-i} \quad (20)$$

where $F(s)$ is obtained in terms of $\delta_+(s)$ and $\lambda_+(s)$. By expanding the exponential in Eq. (18), we can approximate $F(s)$ by

$$F(s)=\frac{\delta_0}{s}+\delta_+(s)+i\delta_0\lambda(s)+i\mathscr{L}\{\delta_+(\tau)\lambda(\tau)\}-\frac{\delta_0}{2}\mathscr{L}\{\lambda^2(\tau)\}+\cdots \quad (21)$$

Equation (20) for the net transverse velocity increment $V-V(0)$ produced by a moment disturbance can now be written as

$$\Delta V=\frac{-L_\theta}{p_0 m}\left[\delta_+(s)+i\delta_0\lambda(s)+i\mathscr{L}\{\delta_+(\tau)\lambda(\tau)\}-\frac{\delta_0}{2}\mathscr{L}\{\lambda^2(\tau)\}+\cdots\right]_{s=-i} \quad (22)$$

where the initial-value contribution from δ_0/s in Eq. (21) is ignored.

Solution for Control

Control feedbacks that minimize dispersion are those that minimize ΔV in Eq. 22. We examine the conditions needed to cause $\Delta V = 0$, which requires

$$\Delta V = (L_\theta / p_0 m)[F(s)]_{s=-i} = 0 \tag{23}$$

or

$$F(-i) = 0 \tag{24}$$

Examined first is the case of a nonzero initial angle of attack δ_0. A first-order approximation to $F(s)$ is

$$F_1(s) = \delta_+(s) + i\delta_0 \lambda(s) \tag{25}$$

which, with Eqs. (12), (13), and (15) and the zero-dispersion condition of Eq. (24), yields the following expression for the feedback gains, for an arbitrary moment disturbance m_t:

$$m_t/\delta_0 = [M]\{[G] + i[H]\}[K] \tag{26}$$

where

$$[M] = [m_y \quad m_z]$$

$$[G] = \frac{(\sigma+1)}{p_0^3\{(\sigma-1)^2 - 4\}} \times \begin{pmatrix} p_0(\sigma-1) & 0 & 0 & -2p_0^2 & 0 & 2 \\ 0 & -p_0(\sigma-1) & -2p_0^2 & 0 & 2 & 0 \end{pmatrix}$$

$$[H] = \frac{(\sigma+1)}{p_0^3\{(\sigma-1)^2 - 4\}} \times \begin{pmatrix} 0 & -2p & -p_0^2(\sigma-1) & 0 & \sigma-1 & 0 \\ -2p_0 & 0 & 0 & p_0^2(\sigma-1) & 0 & -(\sigma-1) \end{pmatrix}$$

$$[K]^T = [A \quad B \quad C \quad D \quad E \quad F] \tag{27}$$

Gains are given in Table 1 for derivative, proportional, and integral control with the disturbance moment about either the y or z body axis and $\delta_0 = i\alpha_0$. Control is feasible with a single feedback variable, and the feedbacks are independent of the type of disturbance (i.e., impulse or step). For an arbitrary moment disturbance with components in both the y and z directions, two nonzero feedbacks are required to satisfy Eq. (26). They depend on the ratio of the directional components of the disturbance.

For the case of zero initial angle of attack, $\delta_0 = 0$, the expression for $F(s)$, Eq. (21) reduces to

$$F(s) = \delta_+(s) + i\mathscr{L}[\delta_+(\tau)\lambda(\tau)] - \tfrac{1}{2}\mathscr{L}[\delta_+(\tau)\lambda^2(\tau)] + \cdots \tag{28}$$

The first-order approximation will no longer hold because the feedbacks do not appear in any linear terms. A second-order approximation to $F(s)$ is

$$F(s) = \delta_+(s) + i\mathscr{L}[\delta_+(\tau)\lambda(\tau)] \tag{29}$$

The solution for zero dispersion is found from Eq. (24). The first term in Eq. (29) is easily evaluated. The higher-order term must be evaluated using a Laplace transform multiplication theorem[6]: If $f_1(t)$ and $f_2(t)$ are $\mathscr{L}$-transformable functions having the $\mathscr{L}$ transforms $F_1(s)$ and $F_2(s)$, respectively, and if $F_1(s) \triangleq A_1(s)/B_1(s)$ is a rational algebraic fraction having q first-order poles and no others, then

$$\mathscr{L}[f_1(t)f_2(t)] = \sum_{k=1}^{q} \frac{A_1(s_k)}{B_1'(s_k)} F_2(s - s_k) \tag{30}$$

By substituting $f_1(t) = \delta_+(\tau)$ and $f_2(t) = \lambda(\tau)$ into Eq. (30) and setting $s = -i$, we can obtain the second-order term.

For an impulsive moment disturbance the solution required for $m_t = m_y$ is

$$m_y\left(2p_0 B - p_0^2 C + \frac{E}{\sigma+1}\right) + ip_0^4[4(\sigma+1) - 1] = 0 \tag{31}$$

and for $m_t = im_z$

$$m_z\left(2p_0 A + p_0^2 D - \frac{F}{\sigma+1}\right) + ip_0^4[4(\sigma+1)] = 0 \tag{32}$$

No solution exists. Addition of the third-order term in $F(s)$ gives the same result. An impulsive moment in uncontrollable by this approach. There is no steady-state angle of attack to provide the lift force required for control.

For a step moment disturbance about the y or z axis, the solution requires one nonzero feedback. The gains are inversely proportional to the magnitude of the disturbance and become quite large for small disturbances. The resulting roll rate excursions violate the perturbation approximation $|p_0| \gg |p_+|$, which suggests that the control is ineffective.

Numerical Examples

The equations of motion, Eqs. (1) and (6), were integrated numerically for both open- and closed-loop responses to impulse and step moment disturbances. Cases for nonzero initial angle of attack were evaluated for the inputs and system parameters shown in Table 2. The gains are summarized in Table 3.

Open- and closed-loop dispersion velocity responses to an impulsive moment about the y axis are shown in Figs. 1 and 2, respectively. The impulse was approximated by a rectangular pulse of 0.0001 s. The magnitude of the disturbance was such that the open-loop angle-of-attack oscillation had an amplitude of roughly 0.1 deg. The closed-loop example is a case with derivative control where $\dot{p} = C\dot{\alpha}$. The open-loop

Table 1 Gains[a] for $\delta_0 = i\alpha_0$

	Control type		
Moment direction	Derivative $\dot{p} = C\dot{\alpha}$	Proportional $\dot{p} = B\beta_+$	Integral $\dot{p} = E\int\alpha_+$
m_y	1	$p_0(\sigma-1)/2$	$-p_0^2$
m_z	$-(\sigma-1)/2$	$-p_0$	$p_0^2(\sigma-1)/2$

[a]Gain $= k \times$ (expression from table), where $k = p_0[(\sigma-1)^2 - 4]/\alpha_0(\sigma^2-1)$.

Table 2 Inputs and system parameters

$m = 22.68$ kg
$L_\theta = 3.275 \times 10^5$ N/rad
$p_0 = 10\pi$ rad/s
$\sigma = 24$
$\mu = 0$
$\nu = 0$
$\Phi(0) = 0$
$v(0) = -40.109$ m/s
$w(0) = 0$
$\alpha(0) = 5$ deg
$\beta(0) = 0$ deg
m_y (steady state) $= 1980.96$ s^{-2}
m_z (steady state) $= 0$

Table 3 Gains for $\alpha_0 = 5$ deg

	A	B	C	D	E	F
m_y	0	118,755	330	0	−324,410	0
m_z	0	−10,325	−3,780	0	3,730,730	0

mean value of the dispersion velocity w is -0.16 m/s. With the closed-loop proportional feedback, the mean value of the dispersion velocity becomes virtually zero.

The response to a step moment resulting in a 0.1-deg step in β is shown in Figs. 3 and 4. The open- and closed-loop cross-plots of the transverse velocities are shown. The mean value of the dispersion velocity is reduced from an open-loop value of 0.8 m/s to a closed-loop value of zero.

Sinusoidal Feedback

The missile response to a sinusoidal roll rate modulation of the form

$$p = p_0 + \Delta \sin qt \tag{33}$$

is examined in order to determine the parameters Δ and q required for effective dispersion control. It is convenient for this analysis to use the aeroballistic equations of motion for the complex angle of attack, Eq. (2), which, with the roll rate behavior, Eq. (33), can be written as

$$\ddot{\xi} + (\nu - i\mu p)\dot{\xi} + (\omega^2 - ip\nu_m)\xi = im_t \exp(i\{p_0 t + \hat{z}[1 - \cos(qt)]\}) \tag{34}$$

where $\hat{z}$ is the nondimensional ratio Δ/q. For Δ small relative to p_0, the roll coupling on the left side of Eq. (34) is very weak because μ and ν_m are, in general, small terms. For a first-order approximation to the roll modulation we assume zero damping $\nu = \nu_m = 0$ and $\mu = 0$, which reduces Eq. (34) to

$$\ddot{\xi} + \omega^2 \xi = im_t \exp(i\{p_0 t + \hat{z}[1 - \cos(qt)]\}) \tag{35}$$

With these approximations it is apparent that the response to an impulsive moment m_t with zero initial angle of attack is not controllable, because the right side of Eq. (35) is defined only at $t = 0$ and is therefore independent of roll rate. Consider the response to a moment step $m_t = -im_z = \bar{m}H(t)$. Equation (35) can be written

$$\ddot{\xi} + \omega^2 \xi = \bar{m}H(t)\, e^{i(\hat{z} + p_0 t)} \{\cos[\hat{z}\cos(qt)] - i \sin[\hat{z}\cos(qt)]\} \tag{36}$$

where the sinusoidal terms can be expanded in terms of Bessel functions with the relations[7]

$$\cos(\hat{z}\cos\theta) = J_0(\hat{z}) + 2\sum_{k=1}^{\infty} (-1)^k J_{2k}(\hat{z})\cos(2k\theta) \tag{37}$$

$$\sin(\hat{z}\cos\theta) = 2\sum_{k=0}^{\infty} (-1)^k J_{2k+1}(\hat{z})\cos[(2k+1)\theta] \tag{38}$$

Equation (36) becomes

$$\begin{aligned}\ddot{\xi} + \omega^2\xi &= \bar{m}H(t)\, e^{i(\hat{z}+p_0 t)}\{J_0(\hat{z}) - 2J_2(\hat{z})\cos(2qt) \\ &\quad + 2J_4(\hat{z})\cos(4qt) - \cdots - i[2J_1(\hat{z})\cos(qt) \\ &\quad - 2J_3(\hat{z})\cos(3qt) + \cdots]\} = \bar{m}H(t)\, e^{i(\hat{z}+p_0 t)} \\ &\quad \times [J_0(\hat{z}) + \cdots + 2(-i)^n J_n(\hat{z})\cos(nqt) + \cdots]\end{aligned} \tag{39}$$

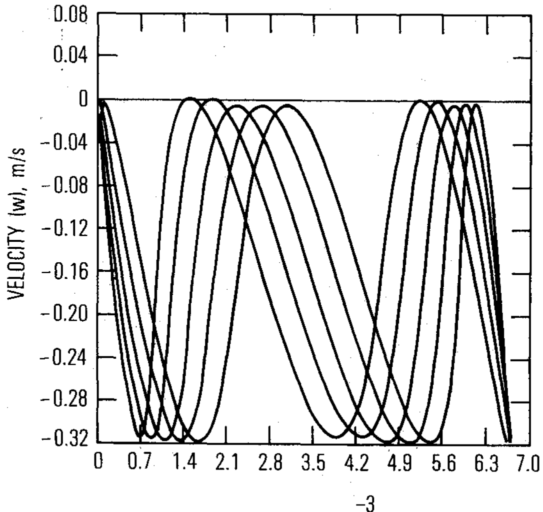

Fig. 1 Open-loop response to moment impulse. Dispersion velocity cross-plot.

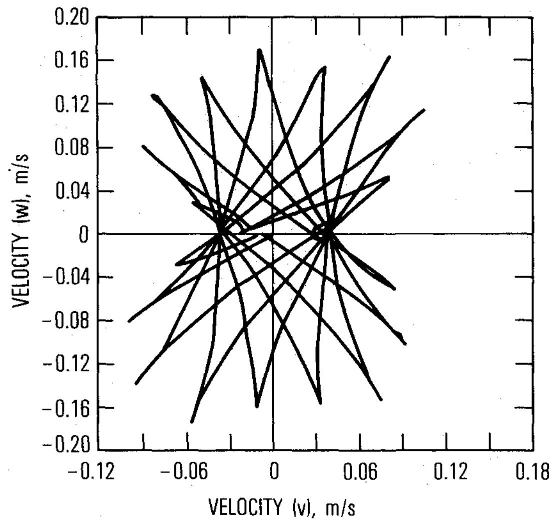

Fig. 2 Closed-loop response to moment impulse. Dispersion velocity cross-plot.

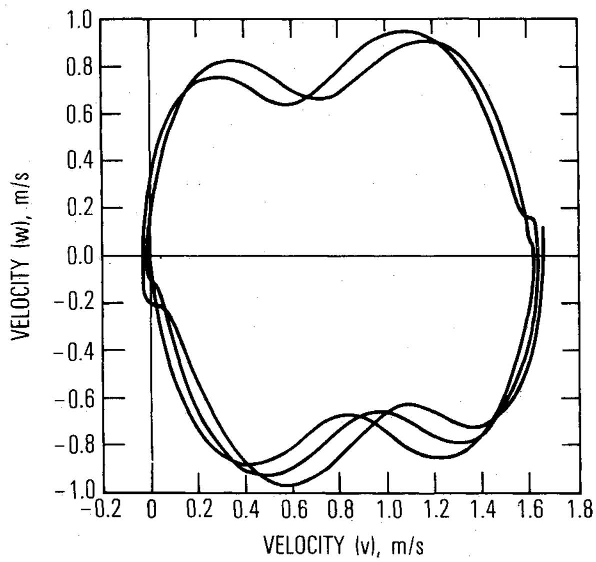

Fig. 3 Open-loop response to moment step. Dispersion velocity cross-plot.

The Laplace transform of Eq. (39) gives

$$\xi(s) = \frac{\overline{m}e^{i\hat{z}}}{s^2+\omega^2}\left[\frac{J_0(\hat{z})}{s-ip_0} + \cdots + \frac{2(-i)^n J_n(\hat{z})(s-ip_0)}{(s-ip_0)^2+(nq)^2} + \cdots\right] \qquad (40)$$

the inverse of which is

$$\xi(t) = \frac{\overline{m}e^{i\hat{z}}J_0(\hat{z})}{\omega^2-p_0^2}\left[e^{ip_0t} - \cos(\omega t) - \frac{ip_0}{\omega}\sin(\omega t)\right] + \cdots + \overline{m}e^{i\hat{z}}J_n(\hat{z})(-i)^n\left\{\frac{1}{\omega^2-(p_0+nq)^2} \times\left[e^{i(p_0+nq)t} - \cos(\omega t) - \frac{i(p_0+nq)}{\omega}\sin(\omega t)\right] + \frac{1}{\omega^2-(p_0-nq)^2}\left[e^{i(p_0-nq)t} - \cos(\omega t) - \frac{i(p_0-nq)}{\omega}\sin(\omega t)\right]\right\} \qquad (41)$$

The transverse velocity, Eq. (7), is obtained by integrating $\xi(t)$ in Eq. (41). The result is

$$V(t) = -\frac{L_\theta \overline{m}e^{i\hat{z}}J_0(\hat{z})}{m(\omega^2-p_0^2)}\left\{\frac{e^{ip_0t}-1}{ip_0} - \frac{\sin(\omega t)}{\omega} - \frac{ip_0}{\omega^2}[1-\cos(\omega t)]\right\} + \cdots + \frac{L_\theta \overline{m}e^{i\hat{z}}J_n(\hat{z})(-i)^n}{m} \times\left(\frac{1}{\omega^2-(p_0+nq)^2}\left\{\frac{e^{i(p_0+nq)t}-1}{i(p_0+nq)} - \frac{\sin(\omega t)}{\omega} - \frac{i(p_0+nq)}{\omega^2}[1-\cos(\omega t)]\right\} + \frac{1}{\omega^2-(p_0-nq)^2} \times\left\{\frac{e^{i(p-nq)t}-1}{i(p_0-nq)} - \frac{\sin(\omega t)}{\omega} - \frac{i(p_0-nq)}{\omega^2}[1-\cos(\omega t)]\right\}\right) \qquad (42)$$

For p_0 equal to integer multiples of q, a singularity occurs at $n = n_s$, where $p_0 - n_s q = 0$. If we let $p_0 - n_s q = \epsilon$, we can obtain the value of the singularity as $\epsilon \to 0$. The singular term in Eq. (42) having $p_0 - nq$ in the denominator can be written

$$\lim_{\epsilon\to 0}\left(\frac{e^{i\epsilon t}-1}{i\epsilon}\right) = \lim_{\epsilon\to 0}\left(\frac{\cos\epsilon t + i\sin\epsilon t - 1}{i\epsilon}\right) \approx t \qquad (43)$$

We are interested in the steady-state, or DC, component of $V(t)$, which is found to be

$$V_{\text{steady state}} = -\frac{iL_\theta\theta_T e^{i\hat{z}}}{mp_0}\left[J_0(\hat{z}) + \cdots + \frac{2(-i)^n J_n(\hat{z})}{1-n^2q^2/p_0^2} + \cdots + (-i)^{n_s}J_{n_s}(\hat{z})(\tfrac{1}{2} - itp_0)\right] \qquad (44)$$

where $\overline{m}$ has been replaced by $\omega^2\theta_t$, θ_t being the magnitude of the trim step. In the limit as Δ and $\hat{z} \to 0$, $J_0(\hat{z}) \to 1$ and $J_n(\hat{z}) \to 0$, and Eq. (44) reduces to the open-loop value for the transverse velocity increment due to a trim step.

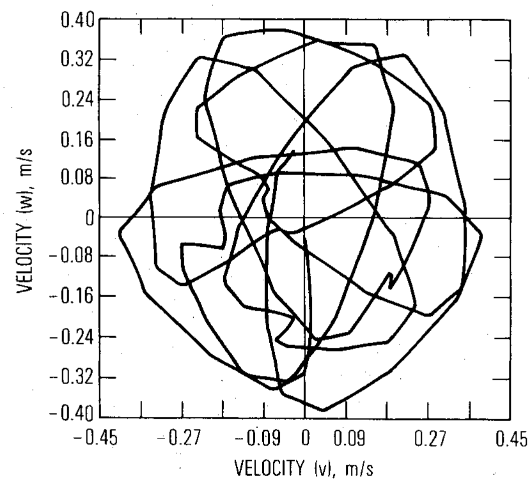

Fig. 4 Closed-loop response to moment step. Dispersion velocity cross-plot.

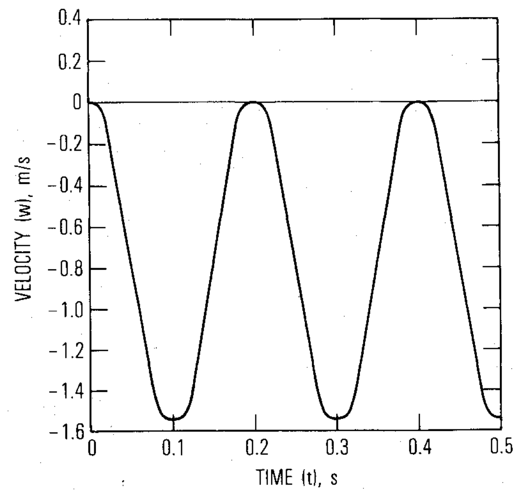

Fig. 5 Open-loop response to moment step. The w component of dispersion velocity.

Numerical Examples

Undamped responses to a step moment with and without the roll modulation were obtained from numerical integration of the equations of motion. The open-loop response of the w component of dispersion velocity is shown in Fig. 5. The mean value, calculated from Eq. (44), with $\hat{z} = 0$ is $w = 0.802$ m/s. The example with roll modulation is the case where $q = p_0$ (Fig. 6). The roll control is

$$\dot{p} = 50\pi\cos 10\pi t \qquad (45)$$

The amplitude is such that the Δ of Eq. (33) is roughly equal to 5. The steady-state response of the w velocity component is predicted from Eq. (44) with $n = 0,1$ and $n_s = (p_0/q) = 1$ to be

$$w = 1.980t - 0.791 \qquad (46)$$

The mean value of w approximated by Eq. (46) agrees well with the value obtained numerically (see Fig. 6) and demonstrates a capability to compensate for lift nonaveraging dispersion. If we consider the amplitude Δ of the roll modulation term to be an equivalent control deflection, we can write a control law of the form

$$\Delta = -Ar - B\dot{r} - C\int r\,\mathrm{d}t \qquad (47)$$

where r is the missile cross-range dispersion. The control loop is shown in Fig. 7. A first-order approximation to the feedback gains can be obtained from the steady-state response to a moment step, Eq. (44). For small values of $\hat{z} = \Delta/q$, the Bessel functions $J_n(\hat{z})$ for $n \geq 1$ are small relative to $J_0(\hat{z}) \approx 1$, and the singularity term dominates the response. For the case $q = p_0$, the steady-state transverse velocity is approximately

$$v + iw = -\frac{iL_\theta \theta_T e^{i\hat{z}}}{mp_0}\left[J_0(\hat{z}) - iJ_1(\hat{z})\left(\tfrac{1}{2} - ip_0 t\right)\right]$$

With $e^{i\hat{z}} \approx 1$, $J_0(\hat{z}) \approx 1$, and $J_1(\hat{z}) \approx \hat{z}/2$, the w component is

$$w = -w_0(1 - t\Delta/2) \tag{48}$$

where $w_0 = L_\theta \theta_T / mp_0$ is the open-loop response. If we let $r = z$ and consider proportional control only, the dispersion z from Eqs. (47) and (48) is described by

$$\dot{z} + w_0 Atz/2 = -w_0 \tag{49}$$

The solution to Eq. (49) is

$$z(t) = e^{-Aw_0 t^2/4}\left[-w_0 \int_0^t e^{Aw_0 t^2/4}\,\mathrm{d}t + z(0)\right] \tag{50}$$

which, with the change of variable $\tau = t\sqrt{Aw_0}/2$, can be written as

$$z(\tau) = -\sqrt{\frac{w_0}{A}}\, e^{-\tau^2}\int_0^\tau e^{\tau^2}\,\mathrm{d}\tau \tag{51}$$

where

$$D(\tau) = e^{-\tau^2}\int_0^\tau e^{\tau^2}\,\mathrm{d}\tau \tag{52}$$

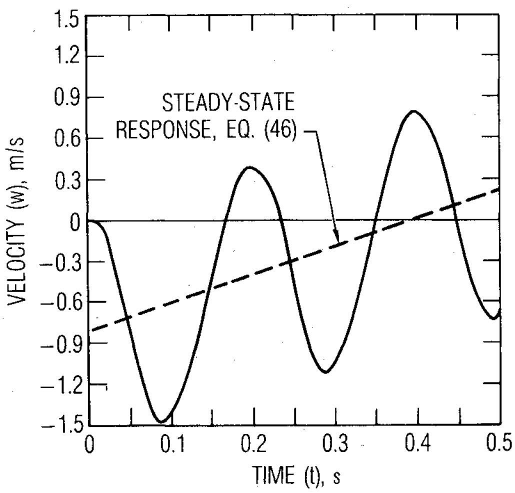

Fig. 6 Roll modulation response to moment step. The w component of dispersion velocity.

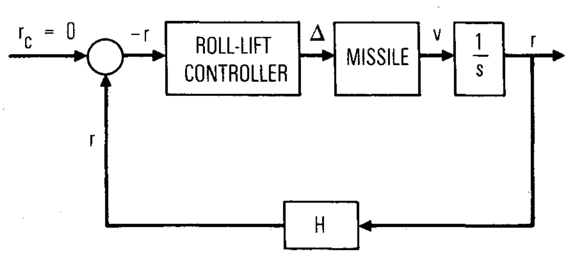

Fig. 7 Roll modulation control loop.

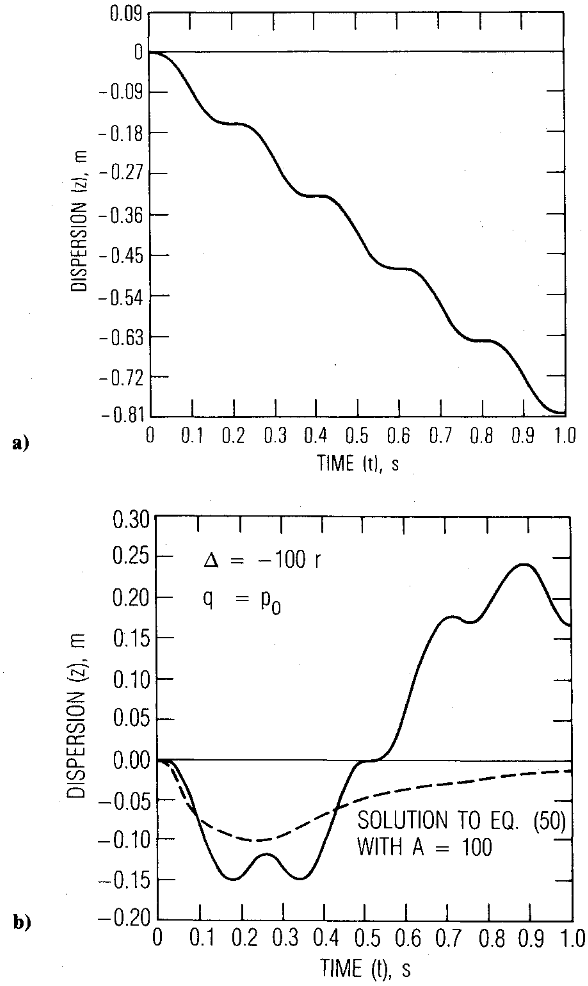

Fig. 8 a) Open-loop response to moment step. The z component of dispersion. b) Closed-loop response to moment step. The z component of dispersion.

The function $D(\tau)$, known as Dawson's integral, is tabulated.[7] We can select a value for the gain A to limit $z(\tau)$ to some prescribed fraction of the open-loop dispersion $w_0 t$. Shown in Fig. 8 is a comparison of the open- and closed-loop response for a gain of $A = 100$, which from Eq. (51) should give a closed-loop response approximately 15% of the open-loop value at $t = 0.45$ s. The undamped oscillation is not an acceptable closed-loop response. Damping should be included also with the feedback $B\dot{r}$.

Conclusions

Cross-range dispersion resulting from lateral moment disturbances can be controlled by modulating the roll rate with the presence of a body-fixed trim. Without an initial trim angle of attack, the dispersion is uncontrollable. Feedbacks that cause the transverse dispersion velocity to average zero can be calculated by linearizing the equations of motion in body-fixed coordinates and obtaining a solution in the Laplace transform domain without the need to invert the transformed equations. A first-order solution is sufficient and shows the gains to be independent of the type of moment disturbance (either impulse or step). An alternative feedback control adjusts the amplitude of a small harmonic modulation of the roll rate about a steady value. The roll modulation frequency required for effective control is derived in the aeroballistic coordinate system, which results in minimal roll coupling with the complex angle-of-attack motion. A quasi-steady solution for dispersion velocity with constant-amplitude roll modulation is then used to derive a control law to limit dispersion. A first-order approximation for the feedback gain produces an undamped response that agrees well with predicted values for the magnitude of dispersion relative to the open-loop response to a moment step. Damping should be included in the feedback to obtain an acceptable closed-loop response.

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